Low complexity networks

6. Sparsity III: Generalized colouring numbers

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Sparsity III: Generalized colouring numbers
Coloring number
Generalized coloring numbers

$$\text{adm}_{r}(G) \leq \text{col}_{r}(G) \leq \text{wcol}_{r}(G) \leq 1 + r(\text{adm}_{r}(G) - 1)r^2$$
## Bounds

<table>
<thead>
<tr>
<th>Class of graphs</th>
<th>$wcol_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bounded expansion</td>
<td>$\leq f(r)$</td>
</tr>
<tr>
<td></td>
<td>(Zhu ’09)</td>
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<tr>
<td>No $K_t$-minor</td>
<td>$(\binom{r+t-2}{t-2})(t-3)(2r+1) \in O(r^{t-1})$</td>
</tr>
<tr>
<td>Planar</td>
<td>$(\binom{r+2}{2})(2r+1) \in O(r^3)$</td>
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</tbody>
</table>

(van den Heuvel, POM, Quiroz, Rabinovich, Siebertz ’17)
Subcoloring

Definition

A *subcoloring* of a graph $G$ is a coloring of the vertices such that each color class induces a disjoint union of cliques.

\[
\max_{H \subseteq_i G} \frac{\chi(H)}{\omega(H)} \leq \chi_{\text{sub}}(G).
\]
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Theorem (Nešetřil, POM, Pilipczuk, Zhu ’19+)

For every graph $G$ and every integer $d \geq 2$ we have

$$\chi_{\text{sub}}(G^d) \leq \begin{cases} 
\text{wcol}_{2d-1}(G) & \text{if } d \text{ is odd,} \\
\text{wcol}_{2d}(G) & \text{if } d \text{ is even.}
\end{cases}$$
Proof

Let $d' = \lfloor d/2 \rfloor$ and
\[
\begin{cases}
(c, <) \text{ a rank } d + 2d' \text{ weak colouring}; \\
v \mapsto \hat{v} := \min \text{ Ball}_{d'}(v); \\
\gamma(v) := c(\hat{v}).
\end{cases}
\]

\[
\begin{cases}
uv \in E(G^d) \\
\gamma(u) = \gamma(v)
\end{cases}
\Rightarrow \hat{u} = \hat{v} \quad \leadsto \quad \text{No } \gamma\text{-monochromatic induced } P_3.
\]
Examples

$$5 \leq \sup_{G \text{ planar}} \chi_{\text{sub}}(G^2) \leq 135.$$
Examples

\[ 5 \leq \sup_{G \text{ planar}} \chi_{\text{sub}}(G^2) \leq 135. \]
Proof that $\chi_{\text{sub}}(G^2) > 4$

- By Ramsey argument we may assume that each level is monochromatic $\rightarrow$ no two consecutive levels have the same color.
Proof that $\chi_{\text{sub}}(G^2) > 4$

- The colors of $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ are neither $\bullet$ nor $\circ$. 

![Graph Diagram]

- The graph $G^2$ has 9 vertices labeled $u, v, a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$. The edges connect vertices in a way that ensures the claimed chromatic number is greater than 4.
Proof that $\chi_{\text{sub}}(G^2) > 4$

- Assume $a_i$ and $b_i$ have the same color.
Proof that $\chi_{\text{sub}}(G^2) > 4$

- Assume $a_i$ and $b_i$ have the same color. Then all the other $a_j$ and $b_j$ have a different color.
Proof that $\chi_{\text{sub}}(G^2) > 4$

- Otherwise, if not all the $a_i$’s have the same color
Proof that $\chi_{\text{sub}}(G^2) > 4$

- So all the $a_i$'s have color ● and all the $b_i$'s have color ●.
Proof that $\chi_{\text{sub}}(G^2) > 4$

- So all the $a_i$’s have color $\bullet$ and all the $b_i$’s have color $\bullet$. By symmetry we can assume that $c_1$ is $\bullet \neq \frac{1}{2}$.
Clustering

Definition

A clustering $\sim$ of a graph $G$ is an equivalence relation on $V(G)$ such that each equivalence class is a clique.

Theorem (Nešetřil, POM, Pilipczuk, Zhu ’19+)

For every integers $d \geq 2$ and $r \geq 1$, and every graph $G$ there exists a clustering $\sim$ of $G$ such that

$$wcol_r(G^d/\sim) \leq wcol_{(d+2\lfloor \frac{d}{2} \rfloor)}(G).$$
Corollary

\[ \max_{H \subseteq iG^d} \frac{\chi(H)}{\omega(H)} \leq \text{wcol}(d+2\lfloor \frac{d}{2} \rfloor)r(G). \]
Consequences

Corollary

\[
\max_{H \subseteq iG^d} \frac{\chi(H)}{\omega(H)} \leq \text{wcol}_{d+2\lfloor d/2 \rfloor}(G).
\]

Corollary

For every \( H \subseteq iG^d \) we have

\[
\frac{\text{col}(H)}{\text{wcol}_{d+2\lfloor d/2 \rfloor}(G)} \leq \omega(H) \leq \chi(H) \leq \text{col}(H).
\]
Using a uniform linear order

**Theorem (van den Heuvel, Kierstead ’19)**

For every graph $G$ there exists a linear order $<^*$ such that for every integer $r$ we have

$$\min_{<^*} \max_{v \in V(G)} |WReach_r(<^*, v)| \leq (2^r + 1) \text{wcol}_{2r}(G)^{4r}.$$ 

**Theorem (Nešetřil, POM, Pilipczuk, Zhu ’19+)**

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Theorem (Nešetřil, POM, Pilipczuk, Zhu ’19+)
For every integer $d \geq 2$ and every graph $G$ there exists a clustering $\sim$ of $G$ such that for every integer $r \geq 1$,

$$\text{wcol}_r(G^d/\sim) \leq (2^{d+2\lfloor d/2 \rfloor})^r + 1) \text{wcol}_{(2d+4\lfloor d/2 \rfloor)r}(G).$$
Consequences

Corollary

If $\mathcal{C}$ has bounded expansion and $d \geq 2$, then there is a clustering $\sim$ such that the class

$$\{G^d/\sim \mid G \in \mathcal{C}\}$$

has bounded expansion.
Distance Coloring of Planar Graphs

Problem

How many colors are needed to ensure that any two vertices at distance 3 get different colors?

\( G^{\#p} \): \( x \) and \( y \) adjacent if \( \text{dist}_G(x, y) = p \).
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$G^{\#p}$: $x$ and $y$ adjacent if $\text{dist}_G(x, y) = p$.

Theorem (Sampathkumar; ’77)

For planar $G$ and every odd $p$ it holds $\chi(G^{\#p}) \leq 5$. 
Distance Coloring of Planar Graphs

Counterexample (Nešetřil, POM)

???

Theorem (Nešetřil, POM; '06)

For all graph $G$ and odd integer $p$ it holds

$$\chi(G\#p) \leq 2^2 \cdot 2^{p\chi(G)}.$$ 

Thus $\sup_{G \in C} \chi(G\#p) < \infty$ for every bounded expansion class $C$. 

$6 \leq \sup_{G \text{ planar}} \chi(G\#3) \leq 5 \cdot 2^{20971522}$
Distance Coloring of Planar Graphs

Counterexample (Nešetřil, POM)

Theorem (Nešetřil, POM; '06)

For all graph $G$ and odd integer $p$ it holds $\chi(G\#^p) \leq 2^{2p} \chi_p(G)^p$.

Thus $\sup_{G \in \mathcal{C}} \chi(G\#^p) < \infty$ for every bounded expansion class $\mathcal{C}$.

$$6 \leq \sup_{G \text{ planar}} \chi(G\#^3) \leq 5 \cdot 2^{20971522}$$
Distance Coloring

Theorem (van den Heuvel, Quiroz, Kierstead 2016)

\[ \chi(G^{\#p}) \leq \begin{cases} \text{wcol}_{2p-1}(G) & \text{if } p \text{ is odd}, \\ \text{wcol}_{2p}(G) \Delta(G) & \text{if } p \text{ is even}. \end{cases} \]

\[ \sup_{G \text{ planar}} \chi(G^{\#(2p+1)}) = O(p^3) \]

\[ 7 \leq \sup_{G \text{ planar}} \chi(G^{\#3}) \leq 103 \]
Odd Distance Coloring

Problem (Van den Heuvel and Naserasr)

Does there exist a constant $C$ such that for every odd-integer $p$ and any planar graph $G$ it holds

$$\chi(G^{#p}) \leq C?$$

Theorem (Bousquet, Esperet, Harutyunyan, de Joannis de Verclos 2018)

$$\chi(G^{#p}) = \Theta\left(\frac{p}{\log p}\right).$$
$r$-neighbourhood covers
$r$-neighbourhood covers

**Theorem (Grohe, Kreutzer, Siebertz 2013)**

For every $\mathcal{C}$ nowhere dense (resp. bounded expansion) class of graphs there is $f$ s.t.
\[ \forall r \in \mathbb{N}, \epsilon > 0, \text{ and } G \in \mathcal{C} \text{ with } |G| \geq f(r, \epsilon) \text{ there exists a family } \mathcal{X} \text{ of subgraphs of } G \text{ s.t.} \]
- the maximum radius of $H \in \mathcal{X}$ is $\leq 2r$;
- every $v \in G$ has all its $r$-neighborhood in some $H \in \mathcal{X}$;
- every $v \in G$ belongs to at most $|G|^\epsilon$ (resp. $K(\mathcal{C}, r, \epsilon)$) subgraphs in $\mathcal{X}$.

**Remark**

Actually a characterization of nowhere dense and bounded expansion monotone classes.
Proof

Consider a linear ordering of $V(G)$ witnessing $\text{wcol}_r(G)$. For $v \in V(G)$, let

$$X_v = \{ u \mid v \in \text{WReach}_r(u) \}.$$
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- Every vertex $u$ belongs to at most $wcol_r(G)$ subsets $X_v$;
Proof

Consider a linear ordering of $V(G)$ witnessing $\text{wcol}_r(G)$.

For $v \in V(G)$, let

$$X_v = \{u \mid v \in \text{WReach}_r(u)\}.$$ 

- Every vertex $u$ belongs to at most $\text{wcol}_r(G)$ subsets $X_v$;
- For every vertex $u$ we have $N_G^r(v) \subseteq X_{\min N_G^r(v)}$. 

$\square$
Exercise

Exercise (Pilipczuk)

For a graph $G$, an integer $r$, and a vertex subset $A \subseteq V(G)$, an $r$-shortest path closure of $A$ is any $B \supseteq A$ such that for all $u, v \in A$ with $\text{dist}_G(u, v) \leq r$, we have

$$\text{dist}_{G[B]}(u, v) = \text{dist}_G(u, v).$$

Prove that for every class $\mathcal{C}$ with bounded expansion and every integer $r$, there exists a constant $c \in \mathbb{N}$, depending on $\mathcal{C}$ and $r$, such that the following holds. For every graph $G \in \mathcal{C}$, one may assign to each vertex $u \in V(G)$ a set $L_u \subseteq V(G)$ with $|L_u| \leq c$, such that for every vertex subset $A \subseteq V(G)$, the set $B := \bigcup_{u \in A} L_u$ is an $r$-shortest path closure of $A$. 