Low complexity networks

6. Sparsity III: Generalized colouring numbers

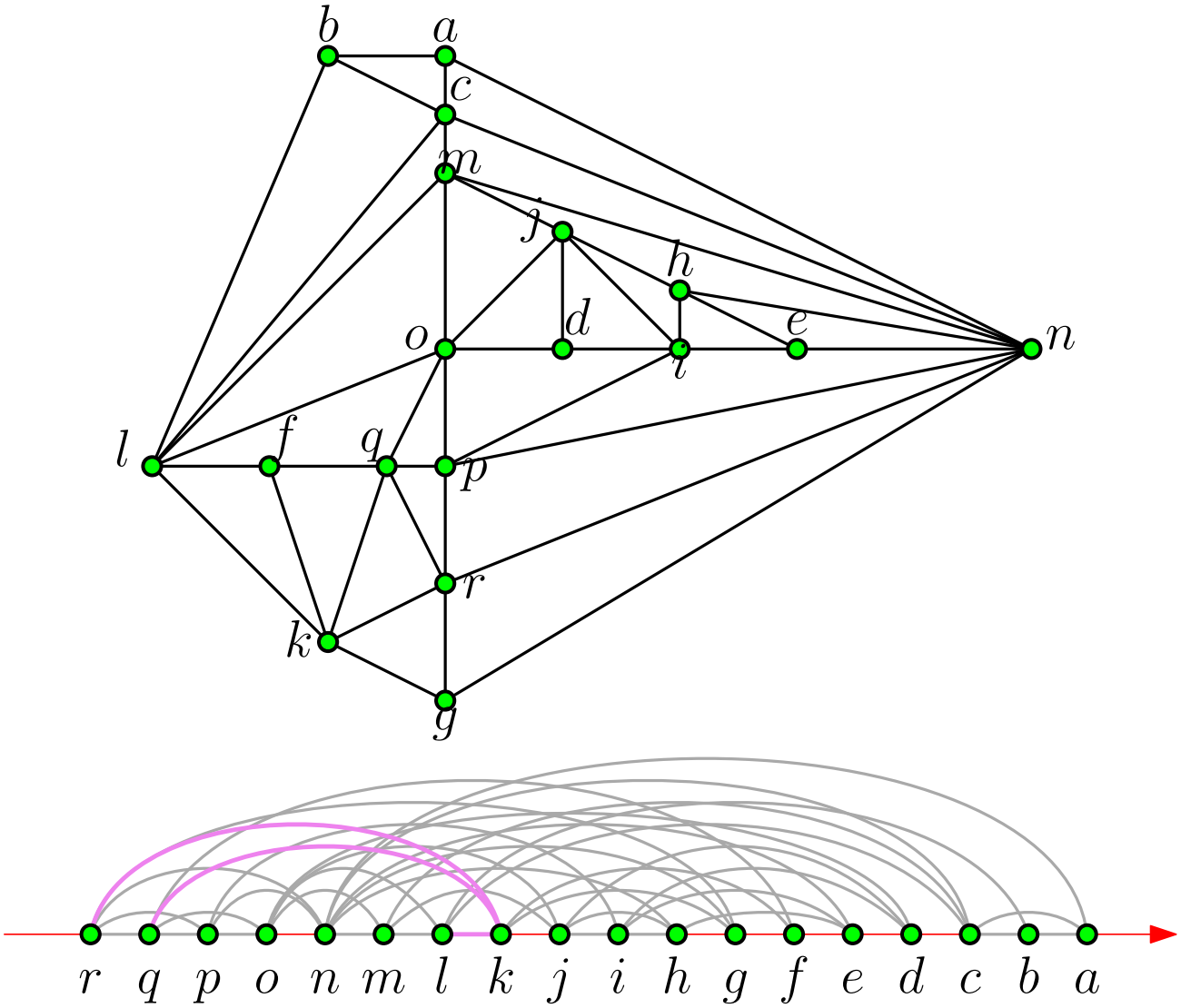
Patrice Ossona de Mendez

Charles University Praha, Czech Republic	CAMS, CNRS/EHESS Paris, France	Zhejiang Normal University Jinhua, China

— IRIF 2020 —

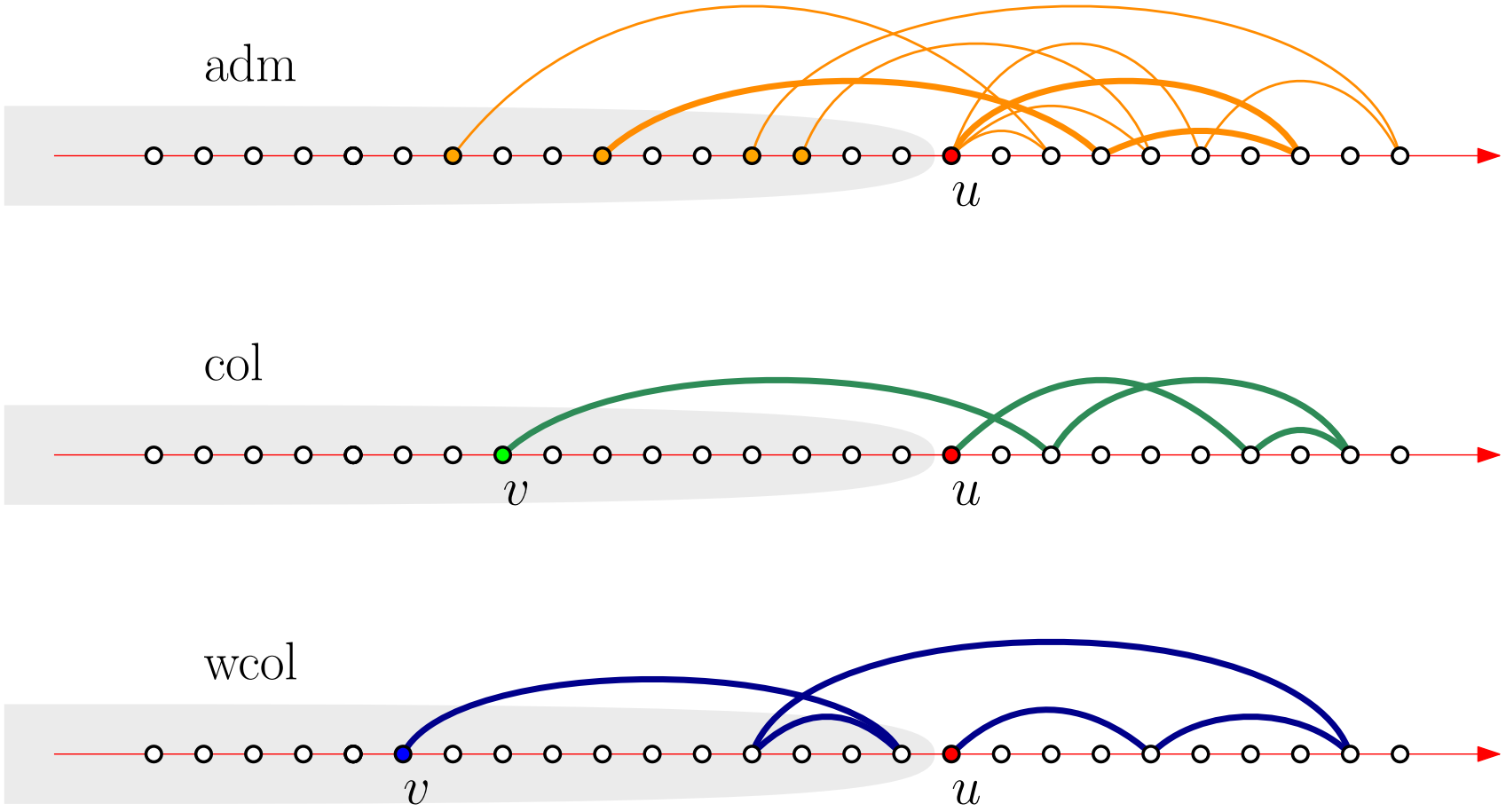
Sparsity III: Generalized colouring numbers

Coloring number



Generalized coloring numbers

$$\mathrm{adm}_r(G) \leq \mathrm{col}_r(G) \leq \mathrm{wcol}_r(G) \leq 1 + r(\mathrm{adm}_r(G) - 1)^{r^2}$$



Bounds

Class of graphs	wcol_r	
Bounded expansion	$\leq f(r)$	(Zhu '09)
No K_t-minor	$\binom{r+t-2}{t-2}(t-3)(2r+1)$	$\in O(r^{t-1})$
Planar	$\binom{r+2}{2}(2r+1)$	$\in O(r^3)$

(van den Heuvel, POM, Quiroz, Rabinovich, Siebertz '17)

Subcoloring

Definition

A *subcoloring* of a graph G is a coloring of the vertices such that each color class induces a disjoint union of cliques.

$$\max_{H \subseteq_i G} \frac{\chi(H)}{\omega(H)} \leq \chi_{\text{sub}}(G).$$

Subcoloring

Definition

A *subcoloring* of a graph G is a coloring of the vertices such that each color class induces a disjoint union of cliques.

$$\max_{H \subseteq_i G} \frac{\chi(H)}{\omega(H)} \leq \chi_{\mathrm{sub}}(G).$$

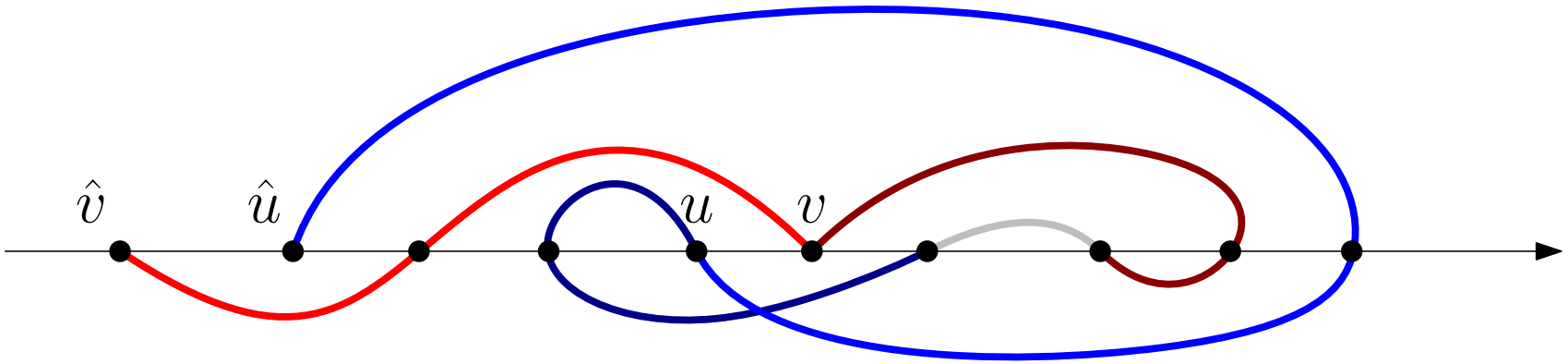
Theorem (Nešetřil, POM, Pilipczuk, Zhu '19+)

For every graph G and every integer $d \geq 2$ we have

$$\chi_{\mathrm{sub}}(G^d) \leq \begin{cases} \mathrm{wcol}_{2d-1}(G) & \text{if } d \text{ is odd,} \\ \mathrm{wcol}_{2d}(G) & \text{if } d \text{ is even.} \end{cases}$$

Proof

Let $d' = \lfloor d/2 \rfloor$ and $\begin{cases} (c, <) \text{ a rank d+2d' weak colouring;} \\ v \mapsto \hat{v} := \min \mathrm{Ball}_{d'}(v); \\ \gamma(v) := c(\hat{v}). \end{cases}$



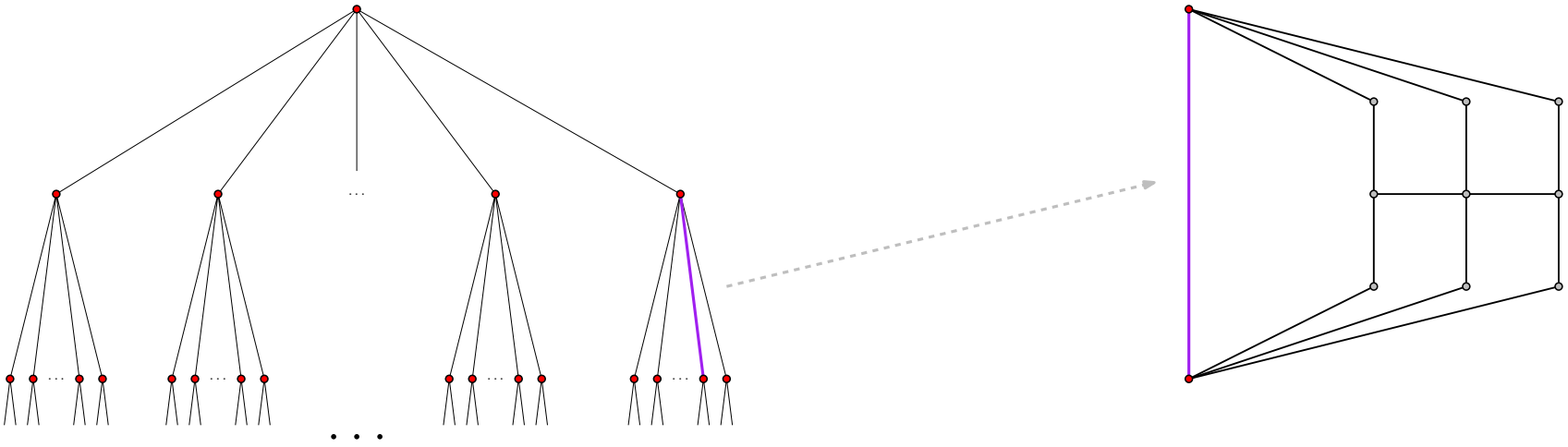
$$\begin{cases} uv \in E(G^d) \\ \gamma(u) = \gamma(v) \end{cases} \Rightarrow \hat{u} = \hat{v} \quad \rightsquigarrow \quad \text{No } \gamma\text{-monochromatic induced } P_3.$$

Examples

$$5 \leq \sup_{G \text{ planar}} \chi_{\text{sub}}(G^2) \leq 135.$$

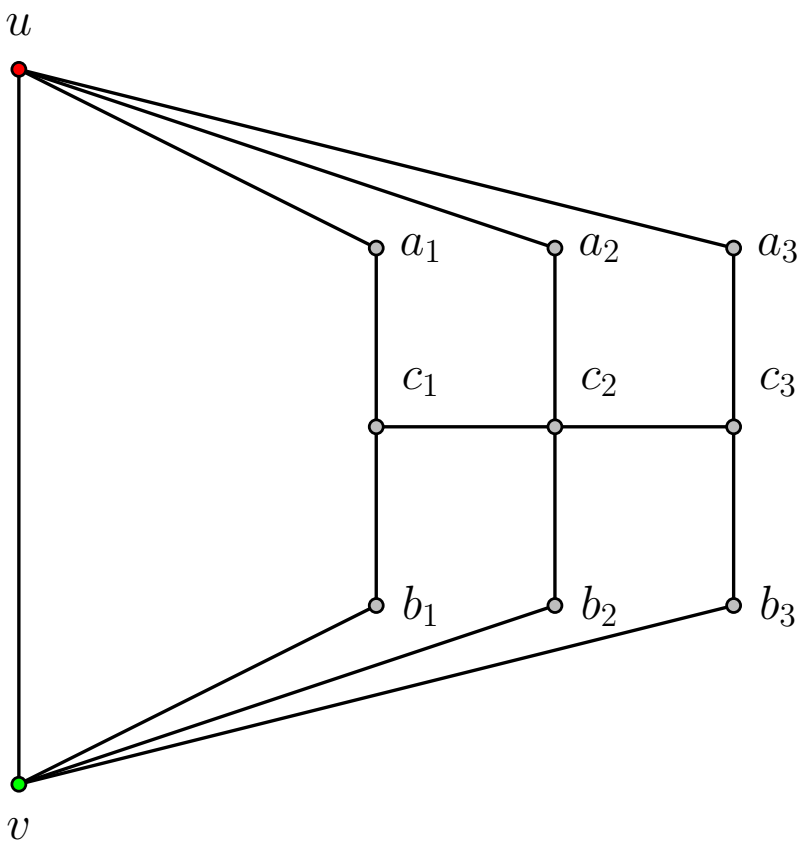
Examples

$$5 \leq \sup_{G \text{ planar}} \chi_{\text{sub}}(G^2) \leq 135.$$



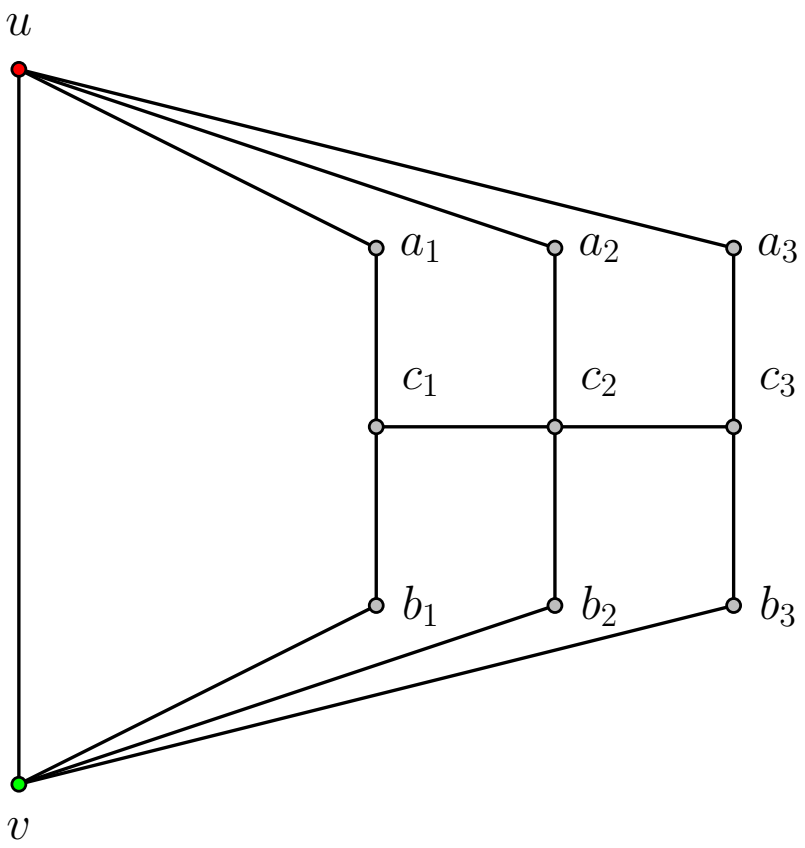
Proof that $\chi_{\text{sub}}(G^2) > 4$

- By Ramsey argument we may assume that each level is monochromatic $\rightarrow$ no two consecutive levels have the same color.



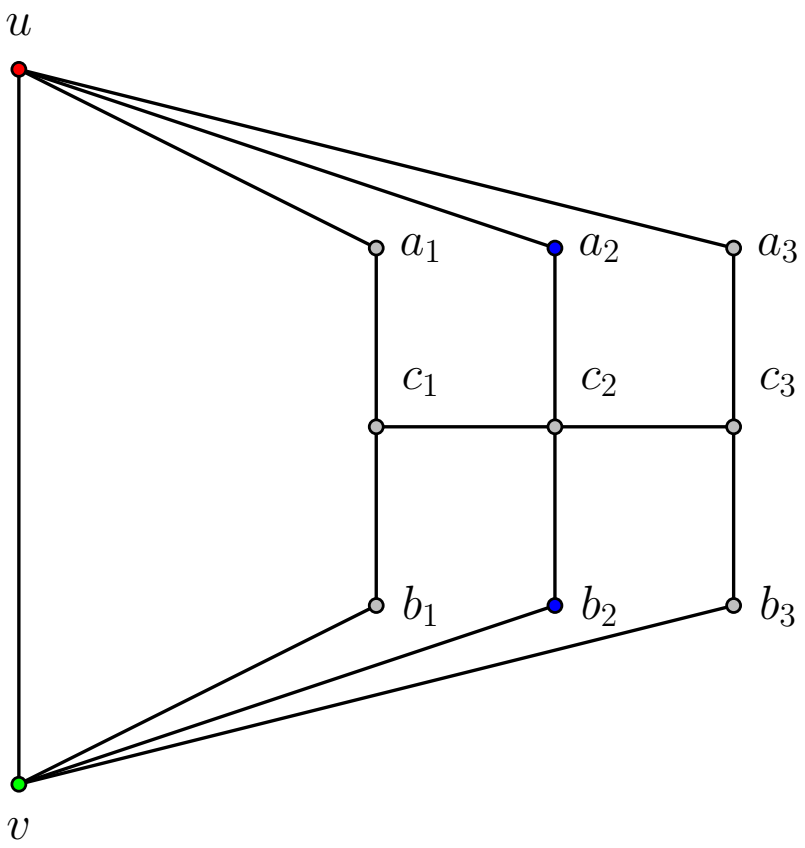
Proof that $\chi_{\text{sub}}(G^2) > 4$

- The colors of $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ are neither ● nor ●.



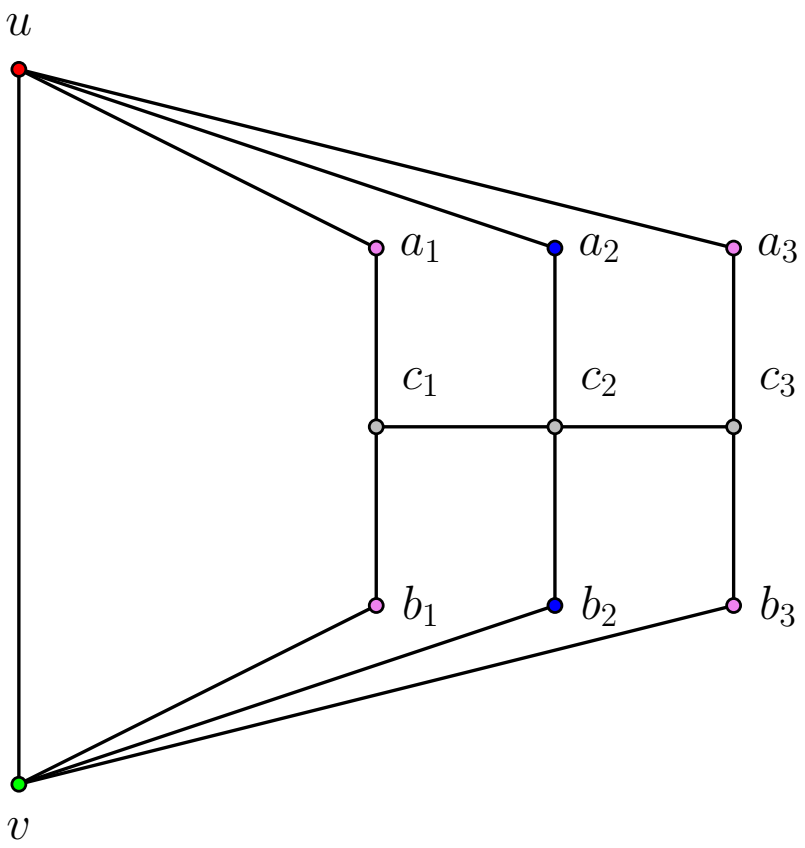
Proof that $\chi_{\text{sub}}(G^2) > 4$

- Assume a_i and b_i have the same color.



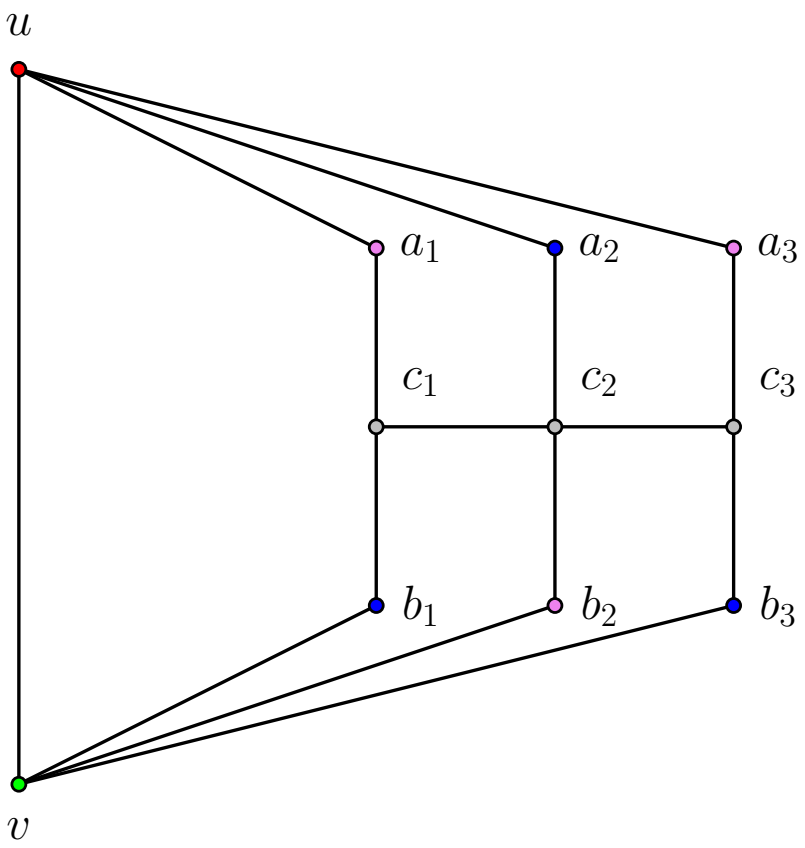
Proof that $\chi_{\text{sub}}(G^2) > 4$

- Assume a_i and b_i have the same color.
  Then all the other a_j and b_j have a different color ↯



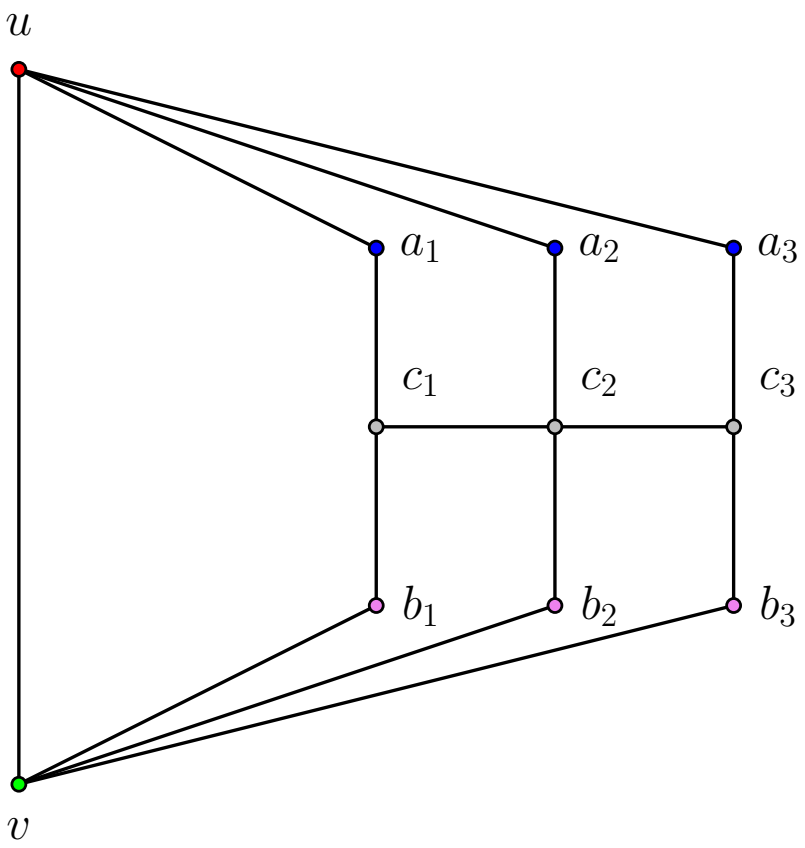
Proof that $\chi_{\text{sub}}(G^2) > 4$

- Otherwise, if not all the a_i's have the same color $\lightning$



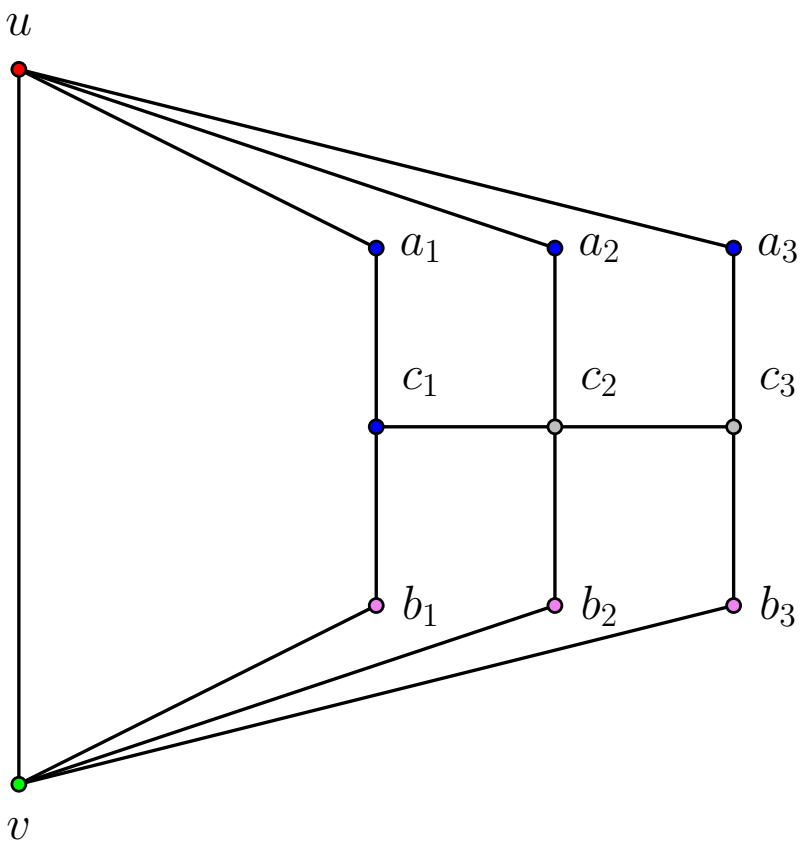
Proof that $\chi_{\text{sub}}(G^2) > 4$

- So all the a_i's have color • and all the b_i's have color •.



Proof that $\chi_{\text{sub}}(G^2) > 4$

- So all the a_i's have color • and all the b_i's have color •. By symmetry we can assume that c_1 is • ↯



Clustering

Definition

A *clustering* $\sim$ of a graph G is an equivalence relation on $V(G)$ such that each equivalence class is a clique.

Theorem (Nešetřil, POM, Pilipczuk, Zhu '19+)

For every integers $d \geq 2$ and $r \geq 1$, and every graph G there exists a clustering $\sim$ of G such that

$$\mathrm{wcol}_r(G^d/\sim) \leq \mathrm{wcol}_{(d+2\lfloor \frac{d}{2} \rfloor)r}(G).$$

Consequences

Corollary

$$\max_{H \subseteq_i G^d} \frac{\chi(H)}{\omega(H)} \leq \mathrm{wcol}_{(d+2\lfloor \frac{d}{2} \rfloor)r}(G).$$

Consequences

Corollary

$$\max_{H \subseteq_i G^d} \frac{\chi(H)}{\omega(H)} \leq \mathrm{wcol}_{(d+2\lfloor \frac{d}{2} \rfloor)r}(G).$$

Corollary

For every $H \subseteq_i G^d$ we have

$$\frac{\mathrm{col}(H)}{\mathrm{wcol}_{d+2\lfloor d/2 \rfloor}(G)} \leq \omega(H) \leq \chi(H) \leq \mathrm{col}(H).$$

Using a uniform linear order

Theorem (van den Heuvel, Kierstead '19)

For every graph G there exists a linear order $<^*$ such that for every integer r we have

$$\min_{<^*} \max_{v \in V(G)} |\mathrm{WReach}_r(<^*, v)| \leq (2^r + 1)\, \mathrm{wcol}_{2r}(G)^{4r}.$$

Theorem (Nešetřil, POM, Pilipczuk, Zhu '19+)

For every integers $d \geq 2$ and $r \geq 1$, and every graph G there exists a clustering $\sim$ of G such that ,

$$\mathrm{wcol}_r(G^d/\sim) \leq \mathrm{wcol}_{(d+2\lfloor \frac{d}{2} \rfloor)r}(G).$$

Using a uniform linear order

Theorem (van den Heuvel, Kierstead '19)

For every graph G there exists a linear order $<^*$ such that for every integer r we have

$$\min_{<^*} \max_{v \in V(G)} |\mathrm{WReach}_r(<^*, v)| \leq (2^r + 1)\, \mathrm{wcol}_{2r}(G)^{4r}.$$

Theorem (Nešetřil, POM, Pilipczuk, Zhu '19+)

For every integer $d \geq 2$ and every graph G there exists a clustering $\sim$ of G such that for every integer $r \geq 1$,

$$\mathrm{wcol}_r(G^d/\sim) \leq (2^{d+2\lfloor d/2 \rfloor)r} + 1)\, \mathrm{wcol}_{(2d+4\lfloor d/2 \rfloor)r}(G).$$

Consequences

Corollary

If $\mathcal{C}$ has bounded expansion and $d \geq 2$, then there is a clustering $\sim$ such that the class

$$\{G^d/\sim \mid G \in \mathcal{C}\}$$

has bounded expansion.

Distance Coloring of Planar Graphs

Problem

How many colors are needed to ensure that any two vertices at distance 3 get different colors?

$G^{\sharp p}$: x and y adjacent if $\mathrm{dist}_G(x, y) = p$.

Distance Coloring of Planar Graphs

Problem

How many colors are needed to ensure that any two vertices at distance 3 get different colors?

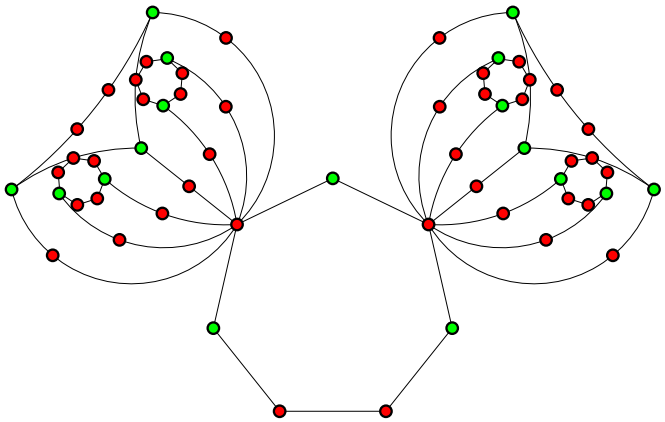
$G^{\sharp p}$: x and y adjacent if $\mathrm{dist}_G(x, y) = p$.

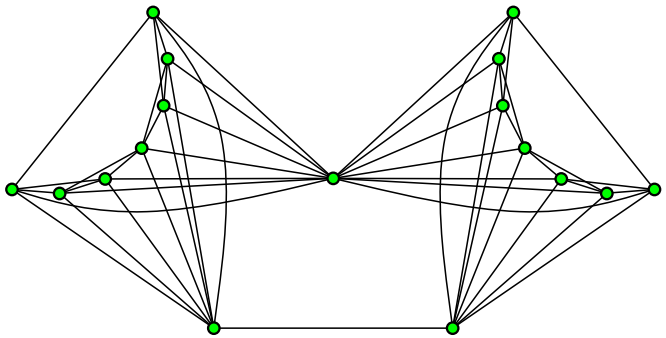
Theorem (Sampathkumar; "77)

For planar G and every odd p it holds $\chi(G^{\sharp p}) \leq 5$.

Distance Coloring of Planar Graphs

Counterexample (Nešetřil, POM)

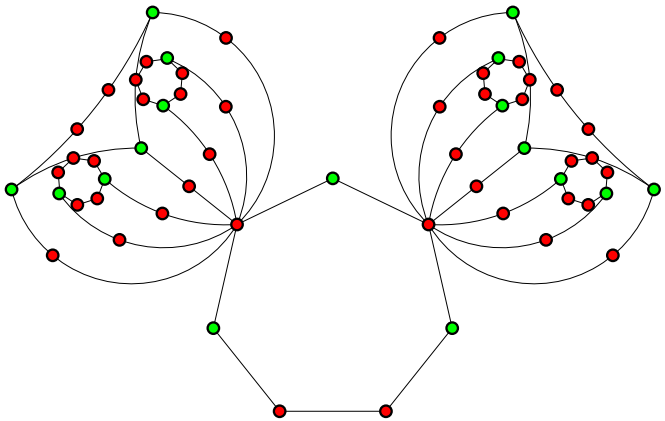


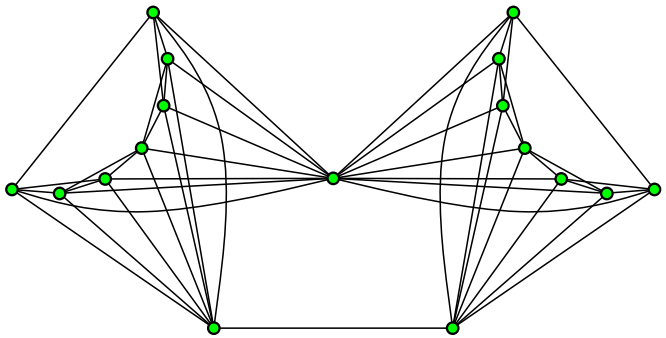


???

Distance Coloring of Planar Graphs

Counterexample (Nešetřil, POM)





Theorem (Nešetřil, POM; '06)

For all graph G and odd integer p it holds $\chi(G^{\sharp p}) \leq 2^{2^{p\,\chi_p(G)^p}}$.
Thus $\sup_{G\in\mathscr{C}} \chi(G^{\sharp p}) < \infty$ for every bounded expansion class $\mathscr{C}$.

$$6 \leq \sup_{G \text{ planar}} \chi(G^{\sharp 3}) \leq 5 \cdot 2^{20971522}$$

Distance Coloring

Theorem (van den Heuvel, Quiroz, Kierstead 2016)

$$\chi(G^{\sharp p}) \leq \begin{cases} \mathrm{wcol}_{2p-1}(G) & \text{if } p \text{ is odd,} \\ \mathrm{wcol}_{2p}(G)\,\Delta(G) & \text{if } p \text{ is even.} \end{cases}$$

$$\sup_{G \text{ planar}} \chi(G^{\sharp(2p+1)}) = O(p^3)$$

$$7 \leq \sup_{G \text{ planar}} \chi(G^{\sharp 3}) \leq 103$$

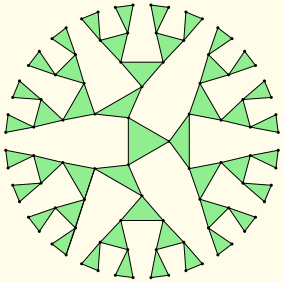
Odd Distance Coloring

Problem (Van den Heuvel and Naserasr)

Does there exist a constant C such that for every odd-integer p and any planar graph G it holds

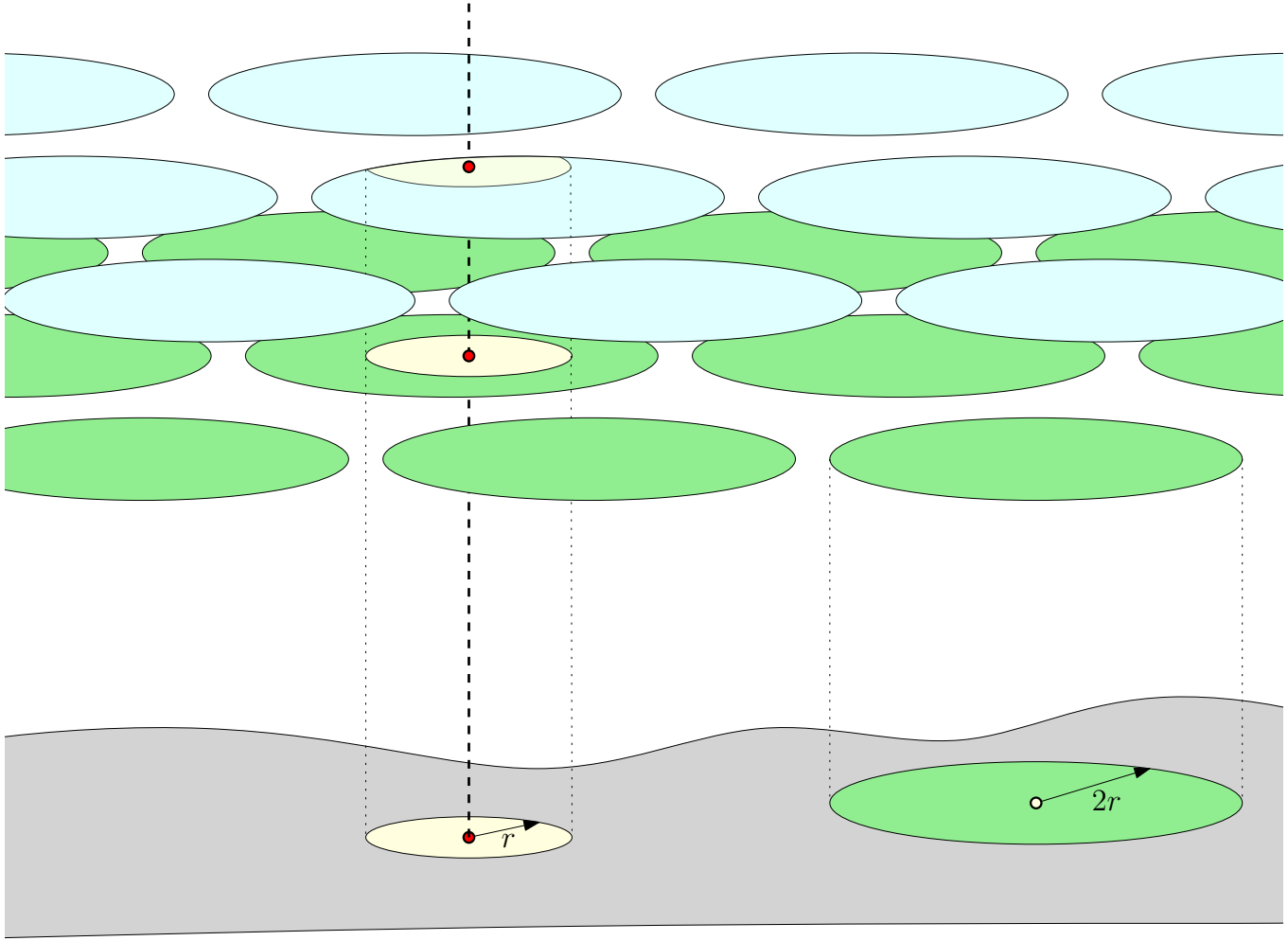
$$\chi(G^{\sharp p}) \leq C?$$

Theorem (Bousquet, Esperet, Harutyunyan, de Joannis de Verclos 2018)



$$\chi(G^{\sharp p}) = \Theta\left(\frac{p}{\log p}\right).$$

r-neighbourhood covers



r-neighbourhood covers

Theorem (Grohe, Kreutzer, Siebertz 2013)

For every $\mathcal{C}$ nowhere dense (resp. bounded expansion) class of graphs there is f s.t.
$\forall r \in \mathbb{N}$, $\epsilon > 0$, and $G \in \mathcal{C}$ with $|G| \geq f(r, \epsilon)$ there exists a family $\mathcal{X}$ of subgraphs of G s.t.

- the maximum radius of $H \in \mathcal{X}$ is $\leq 2r$;
- every $v \in G$ has all its r-neighborhood in some $H \in \mathcal{X}$;
- every $v \in G$ belongs to at most $|G|^{\epsilon}$ (resp. $K(\mathcal{C}, r, \epsilon)$) subgraphs in $\mathcal{X}$.

Remark

Actually a characterization of nowhere dense and bounded expansion monotone classes.

Proof

Consider a linear ordering of $V(G)$ witnessing $\mathrm{wcol}_r(G)$.

For $v \in V(G)$, let

$$X_v = \{u \mid v \in \mathrm{WReach}_r(u)\}.$$

Proof

Consider a linear ordering of $V(G)$ witnessing $\mathrm{wcol}_r(G)$.

For $v \in V(G)$, let

$$X_v = \{u \mid v \in \mathrm{WReach}_r(u)\}.$$

- Every vertex u belongs to at most $\mathrm{wcol}_r(G)$ subsets X_v;

Proof

Consider a linear ordering of $V(G)$ witnessing $\mathrm{wcol}_r(G)$.

For $v \in V(G)$, let

$$X_v = \{u \mid v \in \mathrm{WReach}_r(u)\}.$$

- Every vertex u belongs to at most $\mathrm{wcol}_r(G)$ subsets X_v;
- For every vertex u we have $\mathrm{N}^r_G(v) \subseteq X_{\min \mathrm{N}^r_G(v)}$.

$\square$

Exercise

Exercise (Pilipczuk)

For a graph G, an integer r, and a vertex subset $A \subseteq V(G)$, an *r-shortest path closure* of A is any $B \supseteq A$ such that for all $u, v \in A$ with $\mathrm{dist}_G(u, v) \leq r$, we have

$$\mathrm{dist}_{G[B]}(u, v) = \mathrm{dist}_G(u, v).$$

Prove that for every class $\mathscr{C}$ with bounded expansion and every integer r, there exists a constant $c \in \mathbb{N}$, depending on $\mathscr{C}$ and r, such that the following holds. For every graph $G \in \mathscr{C}$, one may assign to each vertex $u \in V(G)$ a set $L_u \subseteq V(G)$ with $|L_u| \leq c$, such that for every vertex subset $A \subseteq V(G)$, the set $B := \bigcup_{u \in A} L_u$ is an r-shortest path closure of A.