Low complexity networks
5. Sparsity II: Low tree-depth decompositions

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Sparsity II: Low treedepth decompositions
Tree-depth

**Definition**

The *tree-depth* $\text{td}(G)$ of a graph $G$ is the minimum height of a rooted forest $Y$ s.t.

$$G \subseteq \text{Closure}(Y).$$

$$\text{td}(P_n) = \log_2(n + 1)$$
Properties

\[ \text{td}(G) = \begin{cases} \max_H \text{td}(H), & (H \text{ connected component of } G) \\ 1 + \min_v \text{td}(G - v), & (G \text{ connected, } v \text{ vertex of } G) \\ 0, & \text{if } G \text{ is empty} \end{cases} \]

- the tree-depth is minor-monotone:
  \[ H \text{ minor of } G \quad \Rightarrow \quad \text{td}(H) \leq \text{td}(G). \]

- for every graph \( G \) it holds
  \[ \text{tw}(G) \leq \text{td}(G) \leq (\text{tw}(G) + 1) \log_2 |G|. \]
Tree-depth and paths

The tree-depth of a path of order $n$ is $td(P_n) = \lceil \log_2(n + 1) \rceil$. 
The tree-depth of a path of order $n$ is $\text{td}(P_n) = \lceil \log_2(n + 1) \rceil$.

$C$ has bounded tree-depth,

$\iff$ $C$ excludes some path $P_n$ as a subgraph (as a minor),

$\iff$ $C$ is degenerate and excludes some path $P_n$ as an induced subgraph.
A class $\mathcal{T}$ has bounded tree-depth if every DFS-tree has bounded height.
Colored trees

Let \( r_c(n) \) be the number of \( c \)-colored unlabeled rooted trees of order \( n \). Then

\[
    r_c(n) \lesssim \alpha(c)(c\alpha(c)/A)^n n^{-3/2}
\]

Define \( F(c, t) \) by inductively by:

\[
    F(c, t) = \begin{cases} 
    c, & \text{if } t = 1, \\
    \sum_{i=1}^{F(c, t-1)+1} r_c(i), & \text{otherwise.}
    \end{cases}
\]

Lemma

Let \( F \) be a \( c \)-colored rooted forest. If \( \text{height}(F) = t \) and \( |F| > F(c, t) \) then \( F \) has an involutive automorphism exchanging two branches or two rooted trees.
Graphs with bounded tree-depth

**Theorem**

Any $c$-colored graph $G$ of order $n > F(c, \text{td}(G))$ has a non-trivial involuting color-preserving automorphism $\mu : G \rightarrow G$ which reverses no edge.

**Corollary**

Any asymmetric graph of tree-depth $t$ has order at most $F(1, t)$.

**Corollary**

For any $c$-colored graph $G$, $\exists A \subseteq V(G)$, $|A| \leq F(c, t)$, such that $G \rightarrow G[A]$. 
Well quasi orders

Let \((Q, \leq)\) be a well quasi-ordered set and let \(t\) be an integer. Denote by \(\mathcal{T}_t(Q)\) the class of \(Q\)-labeled graphs of tree-depth at most \(t\). Define \(G \subseteq_i H\) if \(\exists f : V(G) \to V(H)\) such that \(G \cong H[f(V(G))]\) and \(\text{label}(f(x)) \geq \text{label}(x)\) for every \(x \in V(G)\).

**Lemma (Ding, ’92)**

The class \(\mathcal{T}_t(Q)\) is well quasi ordered by \(\subseteq_i\).

Remark: the class of 3-colored paths does not have such a property.
\( \chi_p(G) \) is the minimum of colors such that every subset \( I \) of \( \leq p \) colors induces a subgraph \( G_I \) so that \( \text{td}(G_I) \leq |I| \).

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**Theorem (Nešetřil, Ossona de Mendez ’06, ’10)**

\[
\forall p, \sup_{G \in \mathcal{C}} \chi_p(G) < \infty \iff \mathcal{C} \text{ has bounded expansion.}
\]

\[
\forall p, \limsup_{G \in \mathcal{C}} \frac{\log \chi_p(G)}{\log |G|} = 0 \iff \mathcal{C} \text{ is nowhere dense.}
\]

(extends DeVos, Ding, Oporowski, Sanders, Reed, Seymour, Vertigan on low tree-width decomposition of proper minor closed classes, 2004)
## Bounds

<table>
<thead>
<tr>
<th>Class of graphs</th>
<th>$\chi_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bounded degree</td>
<td>$(32\Delta)^2 p$</td>
</tr>
<tr>
<td>Outerplanar</td>
<td>$O(p \log p)$</td>
</tr>
<tr>
<td>Planar</td>
<td>$O(p^3 \log p)$</td>
</tr>
<tr>
<td>Tree-width</td>
<td>$\binom{p+t}{t}$</td>
</tr>
<tr>
<td>No topological $K_t$ minor</td>
<td>$P_t(p)$</td>
</tr>
</tbody>
</table>

(Dębski, Felsner, Micek, Schröder ’20; Pilipczuk, Siebertz ’19)
Application 1: Logarithmic density (again)

Theorem (Nešetřil, Ossona de Mendez)

\[ \forall F : \sup_{t} \limsup_{G \in \mathcal{C} \upharpoonright t} \frac{\log(\#F \subseteq G)}{\log |G|} = |F| \in \{-\infty, 0, 1, \ldots, \alpha(F)\} \]

Somewhere dense

Nowhere dense
$(k, F)$-sunflower $(C, \mathcal{F}_1, \ldots, \mathcal{F}_k)$

$$\forall X_1 \in \mathcal{F}_1, \ldots \forall X_k \in \mathcal{F}_k$$

$$G[C \cup X_1 \cup \cdots \cup X_k] \approx F$$

$$\Rightarrow k \leq \alpha(F) \text{ and }$$

$$(\#F \subseteq G) \geq \prod_{i=1}^{k} |\mathcal{F}_i|.$$
Finding a large sunflower

Let $F$ be a graph of order $p$, let $k \in \mathbb{N}$ and let $0 < \epsilon < 1$. For every graph $G$ such that $(\#F \subseteq G) > |G|^{k+\epsilon}$ there exists in $G$ a $(k+1, F)$-sunflower $(C, \mathcal{F}_1, \ldots, \mathcal{F}_{k+1})$ with

$$\min_i |\mathcal{F}_i| \geq \left( \frac{|G|}{\chi_p(G)^{p/\epsilon}} \right)^{\tau(\epsilon,p)}.$$

Hence $\exists G' \subseteq G$ such that

$$|G'| \geq \left( \frac{|G|}{\chi_p(G')^{p/\epsilon}} \right)^{\tau(\epsilon,p)} \quad \text{and} \quad (\#F \subseteq G') \geq \left( \frac{|G'| - |F|}{k + 1} \right)^{k+1}.$$
Sketch of the Proof

To be proved

Let $F$ be a graph of order $p$, let $k \in \mathbb{N}$ and let $0 < \epsilon < 1$. For every graph $G$ such that $(\#F \subseteq G) > |G|^{k+\epsilon}$ there exists in $G$ a $(k + 1, F)$-sunflower $(C, F_1, \ldots, F_{k+1})$ with

$$\min_i |F_i| \geq \left( \frac{|G|}{c_1(p)(\chi_p(G))} \right)^{c_2(p)\epsilon^p}$$

Steps

- Reduction: general graphs $\rightarrow$ graphs with bounded tree-depth,
- Reduction: graphs with bounded tree-depth $\rightarrow$ colored forests,
- Proof for colored forests.
Reduction to bounded tree-depth

\( \chi_p(G) \) colors \( \rightarrow \) \( \left( \chi_p(G) \right)_p \) subgraphs
LTD

Reduction to bounded tree-depth

\[ \chi_p(G) \text{ colors} \quad \rightarrow \quad \left( \binom{\chi_p(G)}{p} \right) \text{ subgraphs} \]

\[ \implies G \text{ has an induced subgraph } G' \text{ such that } \left( \#F \subseteq G' \right) \geq \left( \#F \subseteq G \right)/\left( \chi_p(G) \right) \text{ and } \text{td}(G') \leq p. \]
Reduction to colored forests

• Color coding
Reduction to colored forests

- Color coding $|F| = p = \#\text{levels} \rightarrow \leq p^p$ possibilities
LTD

Reduction to colored forests

- Color coding \(|F| = p = \#\text{levels} \rightarrow \leq c_1(p)\) possibilities
Reduction to colored forests

- Color coding \(|F| = p = \#\text{levels} \rightarrow \leq c_1(p)\) possibilities
Reduction to colored forests

- Color coding \(|F| = p = \#\text{levels} \rightarrow \leq c_1(p)\)
- Possibilities:
  - \((0,0,1,0,0)\)
  - \((0,1,0,0,0)\)
  - \((0,0,0)\)
  - \((1)\)
  - \(\epsilon\)
  - \((0,0,1,1,1,0)\)
  - \((0,1,0,1,0,1)\)
  - \((1,0,0,0)\)
  - \((0,0,1,0,0)\)
  - \((1,1,0,0,1,0)\)
Sketch of the proof for colored forests

Proof by induction on the height of the forest $F$.

1. Determine where the components of $F$ have to be mapped to get a positive fraction of the copies and some regularity,
2. Partition the components of $F$ depending on the type of images,
3. Select a large “regular” subtree while non decreasing the logarithmic density of copies of $F$, 
4. Use induction to find a $(k + 1, F)$-sunflower
Application 2: Restricted dualities

Problem

How many colors are needed if one requires that every cycle of length $\geq 5$ gets at least 4 colors?

$\Rightarrow \chi_3(G)$ colors are sufficient as $\text{td}(C_k) \geq 4$ if $k \geq 5$.

$\Rightarrow$ bounded for every bounded expansion class.

$\Rightarrow$ Implies that there exists a triangle free graph $H$ such that for every planar graph $G$ it holds

$$K_3 \leftrightarrow G \iff G \rightarrow H$$

This is called a restricted duality.
Restricted dualities

\( C \)-restricted duality

\[
\left\{ \begin{array}{l}
\forall G \in C : \ (\forall F \in \mathcal{F}, \ F \rightarrow G) \iff (G \rightarrow D) \\
\forall F \in \mathcal{F} : \ F \rightarrow D.
\end{array} \right.
\]

Example 1 (Naserasr): \( \forall \) planar \( G \)

Example 2 (Thomassen): \( \forall \) toroidal \( G \)
Classes with all restricted dualities

**Definition**

A class \( \mathcal{C} \) has *all restricted dualities* if every connected \( F \) has a dual \( D \) for \( \mathcal{C} \): \( F \rightarrow D \) and

\[
\forall G \in \mathcal{C}, \quad (F \rightarrow G) \iff (G \rightarrow D).
\]

Theorem (Nešetřil, Ossona de Mendez)

\( \mathcal{C} \) has bounded expansion \( \implies \) \( \mathcal{C} \) has all restricted dualities.
Definition
A class $\mathcal{C}$ has all restricted dualities if every connected $F$ has a dual $D$ for $\mathcal{C}$: $F \nrightarrow D$ and

$$\forall G \in \mathcal{C}, \quad (F \rightarrow G) \iff (G \rightarrow D).$$

Theorem (Nešetřil, Ossona de Mendez)
$\mathcal{C}$ has bounded expansion $\implies$ $\mathcal{C}$ has all restricted dualities.
Coloring random graphs

Theorem (Achlioptas, ’04)
Let $k_d$ be the smallest integer $k$ such that $d < 2k \log k$. Then $G(n, d/n)$ has chromatic number either $k_d$ or $k_d + 1$ a.a.s.

Let $d_3$ be the threshold for 3-colorability. Then

- $d_3 \lesssim 4.85$ (Dubois, Mandler ’03)
- $d_3 \gtrsim 4.03$ (Achlioptas, Moore ’03)

Problem (Spencer)
Assume $d_3$ is such that 3-coloring is a non-trivial problem on $G(n, d_3/n)$.
Does there exist a polynomial time to decide 3-coloring on $G(n, d_3/n)$ a.a.s.?
Problem

What about other $H$-coloring problems?
Are there any easy non-trivial $H$-coloring problems?
$H$-coloring random graphs

**Problem**

What about other $H$-coloring problems?
Are there any easy non-trivial $H$-coloring problems?

**Restricted dualities!**

**Theorem (Nešetřil, Ossona de Mendez ’17+)**

For every $d > 0$, and every odd integer $p \geq 5$ there is a graph $D_{d,p}$ such that
- odd-girth($D_{d,p}$) $\geq p$,
- $D_{d,p}$-colorability is non-trivial on $G(n, d/n)$,
- $D_{d,p}$-colorability can be decided a.a.s. on $G(n, d/n)$ in linear time (with certificates).
Conjecture

Let $H$-coloring be a non trivial problem for $G(n, d/n)$. The following are equivalent:

- for every $\epsilon > 0$ there is a polynomial time decidability algorithm for $H$-coloring with error at most $\epsilon$,
- there is a linear time a.a.s. decidability algorithm for $H$-coloring,
- either $H = K_2$ or $H$ is a restricted dual of some $C_p$ for odd $p$ in a class $\mathcal{C}$ with measure 1 in $G(n, d/n)$;
- $H$-coloring has a coarse threshold.
Thresholds

Definition

The threshold $p_0(n)$ for property $\mathcal{P}$ is \textit{sharp} if $\forall \epsilon > 0$

$$\Pr[G(n, p(n)) \in \mathcal{P}] \rightarrow \begin{cases} 0 & \text{if } p(n) \leq (1 - \epsilon)p_0(n) \\ 1 & \text{if } p(n) \geq (1 + \epsilon)p_0(n) \end{cases}$$

Otherwise it is \textit{coarse}

Example

- connectivity, $k$-colorability ($k > 2$): \textit{sharp};
- contains a triangle: \textit{coarse}. 
Exercise (Pilipczuk)

Prove that there exists an algorithm that, given an \( n \)-vertex graph \( G \) together with its tree-depth decomposition of height at most \( d \), verifies whether \( G \) admits a proper 3-coloring in time \( O(3^d n^c) \), for some constant \( c \) independent of \( d \). The constants hidden in the \( O(\cdot) \)-notation may not depend on \( d \).