Minors

Low complexity networks

4. Sparsity I: Measuring shallow minors

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Sparsity I: Measuring shallow Minors
Many kind of minors

<table>
<thead>
<tr>
<th>Minor</th>
<th>Topological minor</th>
<th>Immersion</th>
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<tbody>
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<td>$H \leq_m G$</td>
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- $H \leq_m G$: Minor
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Minors

- $G \leq_m H$: minor relation
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  - well quasi order (Robertson, Seymour; ’04)

- $G \leq_t H$: topological minor relation
  - not a well quasi order
  - Hajós' conjecture (false for almost all graphs, but true if large girth)

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  - conjecture of Abu-Khzam and Langston — Lescure and Meyniel (proved for $k \leq 7$, DeVos, Kawarabayashi, Mohar, and Okamura ’09)
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Minors

Minors and minimum degree

Theorem (Komlós and Szemerédi ’94, Bollobás and Thomason ’98)

There exists a constant $c$ such that every graph $G$ with minimum degree at least $ck^2$ is such that $K_k \leq t G$. 

Theorem (Kostochka ’82, Thomason ’84)

There exists a constant $\gamma \approx 0.319$ such that every graph $G$ with minimum degree at least $\gamma k \sqrt{\log(k)}$ is such that $K_k \leq m_G$.

Theorem (Dvořák, Yepremyan ’18)

If a simple graph $G$ has minimum degree $11k + 7$ then $K_k \leq i_G$. 
Minors and minimum degree

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Shallow minors

**Minor** 

$G \triangleright t$ \supseteq \nrightarrow t$

**Topological minor** 

$G \tilde{\triangleright} t$ \subseteq \nrightarrow 2t$

**Immersion** 

$G \overset{\tilde{\triangleright}}{\supseteq} t$ \subseteq \nrightarrow 2t

\n
\[ G \triangleright t \supseteq \mathcal{C} \triangleright t \]

\[ G \tilde{\triangleright} t \subseteq \mathcal{C} \tilde{\triangleright} t \]

\[ G \overset{\tilde{\triangleright}}{\supseteq} t \subseteq \mathcal{C} \overset{\tilde{\triangleright}}{\supseteq} t \]
Graph invariants

\[ \text{mad}(G) + 1 \geq \text{col}(G) \geq \chi(G) \geq \chi_f(G) \geq \rho(G) \geq \omega(G). \]

Maximum average degree \( \text{mad}(G) = \max_{H \subseteq G} \overline{d}(H) \)

Coloring number \( \text{col}(G) = 1 + \max_{H \subseteq G} \delta(H) \)

Chromatic number \( \chi(G) \)

Fractional chromatic number \( \chi_f(G) = \inf \left\{ \frac{a}{b} \left| G \rightarrow KG_{a,b} \right. \right\} \)

Hall ratio \( \rho(G) = \max_{H \subseteq G} |H|/\alpha(H) \)

Clique number \( \omega(G) \)
Density and chromatic number of 1-subdivisions

Lemma (Dvořák, ’07)

Let \( c \geq 4 \) be an integer and let \( G \) be a graph with minimum degree \( d > 56(c - 1)^2 \frac{\log(c-1)}{\log c - \log(c-1)} \). Then the graph \( G \) contains a subgraph \( G' \) that is the 1-subdivision of a graph with chromatic number \( c \).
Density and chromatic number of 1-subdivisions

**Lemma (Dvořák, ’07)**

Let $c \geq 4$ be an integer and let $G$ be a graph with minimum degree $d > 56(c - 1)^2 \frac{\log(c-1)}{\log c - \log(c-1)}$. Then the graph $G$ contains a subgraph $G'$ that is the 1-subdivision of a graph with chromatic number $c$.

**Theorem (Dvořák, Ossona de Mendez, Wu ’19+)**

For every $c \geq 10$, every graph of average degree at least $256c^3$ contains the 1-subdivision of a graph of fractional chromatic number at least $c$. 
Class Taxonomy

**Definition**

A class $\mathcal{C}$ has **bounded expansion** if there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that

$$\forall r \in \mathbb{N} \quad \sup \{ \bar{d}(G) \mid G \in \mathcal{C} \triangleleft r \} \leq f(r).$$

**Definition**

A class $\mathcal{C}$ is **nowhere dense** if there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that

$$\forall r \in \mathbb{N} \quad \sup \{ \omega(G) \mid G \in \mathcal{C} \lhd r \} \leq f(r).$$
Examples

• Graphs with maximum degree 100

• Planar graphs

• Random graphs $G(n, d/n)$
Examples

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  Bounded expansion: $\bar{d}(C \ominus t) < 100^t$

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Examples

- Graphs with maximum degree 100
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- Planar graphs
  Bounded expansion: $\overline{d}(C \triangleright t) < 6$ (Euler)

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• Graphs with maximum degree 100
  Bounded expansion: $\overline{d}(\mathcal{C} \triangle t) < 100^t$

• Planar graphs
  Bounded expansion: $\overline{d}(\mathcal{C} \triangle t) < 6$ (Euler)

• Random graphs $G(n, d/n)$
  $\exists$ bounded expansion class $\mathcal{R}_d$ s.t. $G(n, d/n) \in \mathcal{R}_d$ a.a.s.
## Class Taxonomy

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More Examples

• Class of $G$ without cycles of length $\leq 10^{10^{10}}$

• Class of $G$ such that $\Delta(G) \leq f(\text{girth}(G))$

• Class of $G$ with $\|G\| > |G|^{1+\epsilon}$
More Examples

- Class of $G$ without cycles of length $\leq 10^{10^{10}}$
  Somewhere dense: $10^{10^{10}}$-subdivisions of $K_n$

- Class of $G$ such that $\Delta(G) \leq f(girth(G))$

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More Examples

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  Somewhere dense: $10^{10^{10}}$-subdivisions of $K_n$

- Class of $G$ such that $\Delta(G) \leq f(\text{girth}(G))$
  Nowhere dense: $\omega(G \overline{\Delta} t) \leq f(6t)$

- Class of $G$ with $\|G\| > |G|^{1+\epsilon}$
More Examples

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  Nowhere dense: $\omega(G \tilde{\nabla} t) \leq f(6t)$

• Class of $G$ with $\|G\| > |G|^{1+\epsilon}$
  Somewhere dense: $\supseteq 8/\epsilon$-subdivisions of $K_n$
Erdős-Rényi random graphs

Theorem (Nešetril, Ossona de Mendez, Wood ’12; Nešetril, Ossona de Mendez ’17+)

• If \( p(n) \approx d/n \), there exists a bounded expansion class \( R_d \) such that \( G(n, p(n)) \in R_d \) a.a.s.

• If \( p(n) \ll n^{-1+\epsilon} \) for every \( \epsilon > 0 \), there exists a nowhere dense class \( C \) such that \( G(n, p(n)) \in C \) a.a.s.

• If \( p(n) \gtrsim n^{-1+\epsilon} \) (for some \( \epsilon > 0 \)), then \( G(n, p(n)) \) a.a.s. contains an \( O(1/\epsilon) \)-subdivision of arbitrarily large complete graphs.
• **Configuration Model** and the **Chung-Lu Model** with specified asymptotic degree sequences

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<tr>
<th>Distribution</th>
<th>Degree Sequence</th>
<th>Constraints</th>
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<td>Power law</td>
<td>$d^{-\gamma}$</td>
<td>$\gamma &gt; 2$</td>
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<td>Power law w/ cutoff</td>
<td>$d^{-\gamma}e^{-\lambda d}$</td>
<td>$\gamma &gt; 2, \lambda &gt; 0$</td>
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<td>Exponential</td>
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<tr>
<td>Stretched exponential</td>
<td>$d^{\beta-1}e^{-\lambda d\beta}$</td>
<td>$\lambda, \beta &gt; 0$</td>
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<tr>
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<td>$d^{-1}\exp(-\frac{(\log d-\mu)^2}{2\sigma^2})$</td>
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• generalization of **Erdős-Rényi graphs** (perturbed bounded-degree graphs), which includes the stochastic block model with small probabilities.
Unavoidable subgraphs

\textbf{Theorem (Erdős, Simonovits, Stone)}

\[
\text{ex}(n, H) = \left( 1 - \frac{1}{\chi(H) - 1} \right) \binom{n}{2} + o(n^2).
\]
Unavoidable subgraphs

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Theorem (Jiang, Seiver ’12)

Let $F$ be a subdivision of a graph $H$, where each edge is subdivided by an even number of vertices (at least $2m$). Then

$$\text{ex}(n, F) = O(n^{1+\frac{8}{m}}).$$
Minors

Concentration

Theorem (Jiang, Seiver ’12)

\[ \text{ex}(n, K_t^{(\leq p)}) = O(n^{1+\frac{8}{p}}). \]

\[ C \subseteq C \gtrsim 0 \subseteq \ldots \subseteq C \gtrsim t \subseteq \ldots \subseteq C \gtrsim \frac{8t}{\epsilon} \subseteq \ldots \subseteq C \gtrsim \infty \]

\[ \|G\| > C_t |G|^{1+\epsilon} \]

Hence:

\[ \limsup_{G \in C \gtrsim t} \frac{\log \|G\|}{\log |G|} > 1 + \epsilon \implies \limsup_{G \in C \gtrsim \frac{8t}{\epsilon}} \frac{\log \|G\|}{\log |G|} = 2. \]
Classification by logarithmic density

Theorem (Nešetřil, Ossona de Mendez)

Let $\mathcal{C}$ be an infinite class of graphs. Then

$$\sup_t \limsup_{G \in \mathcal{C}} \frac{\log \|G\|}{\log |G|} \in \{-\infty, 0, 1, 2\}.$$ 

• **bounded size class** $\iff -\infty$ or $0$;  
• **nowhere dense class** $\iff -\infty, 0$ or $1$;  
• **somewhere dense class** $\iff 2$. 

Minors
Exercise

Prove that a class $\mathcal{C}$ has bounded expansion if and only if there exists a function $f$ such that

$$\forall r \in \mathbb{N} \ \forall G \in \mathcal{C} \setminus r \quad \chi_f(G) \leq f(r).$$