Low complexity networks

2. First-order Logic

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First-order logic
Relational Structures

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- First order logic generalize subgraphs, homomorphisms from a template,...
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- First order logic generalize subgraphs, homomorphisms from a template, ...
  → local properties
- Monadic second order logic generalize minors, colorings, homomorphisms to a template, ...
  → global properties
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- **Relational vocabulary** $\sigma$: finite set of symbols with arity. A $\sigma$-structure $A$: a universe (or domain) $A$, and an interpretation $R \in \sigma \mapsto R^A \subseteq A^r$. 
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- $B$ is a *substructure* of $A$ if $B \subseteq A$ and $R^B \subseteq R^A$;
  
  $B$ is an *induced substructure* of $A$ if $R^B = R^A \cap B^r$. 

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- $B$ is a substructure of $A$ if $B \subseteq A$ and $R^B \subseteq R^A$; $B$ is an induced substructure of $A$ if $R^B = R^A \cap B^r$.

- A homomorphism $A \to B$ is a mapping $f : A \to B$ such that:

  $$(x_1, \ldots, x_k) \in R^A \quad \Rightarrow \quad (f(x_1), \ldots, f(x_k)) \in R^B.$$ 

  The class (category) of all finite $\sigma$-structures is denoted by $\text{Rel}(\sigma)$, at most countable $\sigma$-structures by $\text{Rel}_\omega(\sigma)$. 
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- The *quantifier count* $\text{qcount}(\phi)$ of $\phi$ is the total number of quantifiers in $\phi$.

- The *quantifier rank* $\text{qrank}(\phi)$ of $\phi$ is the maximum nesting of quantifiers of its sub-formulas.
• **atomic formulas, Boolean formulas, existential first-order formulas, first-order formulas.**

• The **quantifier count** $q\text{count}(\phi)$ of $\phi$ is the total number of quantifiers in $\phi$.

• The **quantifier rank** $q\text{rank}(\phi)$ of $\phi$ is the maximum nesting of quantifiers of its sub-formulas.

For a formula $\phi[x_1, \ldots, x_n]$ with free variables $x_1, \ldots, x_n$,

$$A \models \phi[a_1, \ldots, a_n] \iff \phi \text{ is true in } A \text{ when } x_i \leftarrow a_i.$$
Example: $\operatorname{dist}(x, y) \leq d$

- **Naive approach**

  $$(\exists v_0, \ldots, v_d) \ (x = v_0) \land \left[ \bigwedge_{i=0}^{d-1} E(v_i, v_{i+1}) \lor (v_i = v_{i+1}) \right] \land (v_d = y).$$

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- **Binary search approach**

  $$\delta_d(x, y) := \begin{cases} E(x, y) \lor (x = y) & \text{if } d = 1, \\ (\exists z) \ \delta_{\lfloor d/2 \rfloor}(x, z) \land \delta_{\lceil d/2 \rceil}(z, y) & \text{otherwise.} \end{cases}$$

  $\rightarrow$ qrank $\approx \log d$. 
Theories and Models

Definition

A *theory* $T$ is a set of sentences,

a *model* $M$ of a theory $T$ is a structure where all the sentences in $T$ are satisfied: $M \models T$.

Theorem (Gödel completeness theorem)

A sentence $\theta$ can be proved in a theory $T$ (i.e. $T \vdash \theta$) if and only if every model of $T$ satisfies $\theta$ (i.e. $T \models \theta$).

Theorem (Henkin)

Every consistent theory has a model.
0 \mathbf{-} 1 \text{ law}

Definition

$G(n, p(n))$ satisfies a 0 \mathbf{-} 1 \text{ law} \text{ (for first-order logic)} if every first-order property is either false or true a.a.s.
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Theorem (Shelah, Spencer ’88)

If \( p(n) = n^{-\alpha} \) then \( G(n, p(n)) \) satisfies a 0 – 1 law for first-order logic if and only if \( \alpha \) is an irrational number.
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  - In the first case, the Duplicator chooses a vertex $b$ in $H$;
  - If no isomorphism $\pi_i : G[A_i] \rightarrow G[B_i]$ extending $\pi_{i-1}$ exists such that $\pi_i(a) = b$ then Spoiler wins the game.
  - If there is an isomorphism $\pi_i : G[A_i] \rightarrow G[B_i]$ extending $\pi_{i-1}$ then the game continues until $i = n$. If $i$ reaches $n$ and $\pi_n$ is an isomorphism from $G[A_n]$ to $H[B_n]$ then Duplicator wins the game.
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- Let \( A_i = A_{i-1} \cup \{a\} \) and \( B_i = B_{i-1} \cup \{b\} \);
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Back and Forth Equivalence

If Duplicator has a winning strategy for $n$ then $G$ and $H$ are $n$-back and forth equivalent.
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Two graphs (and more generally two structures) are $n$-back and forth equivalent if and only if they satisfy the same first order sentences of quantifier rank $n$ (denoted by $G \equiv_n H$).
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**Definition**

Two \( \sigma \)-structures \( A \) and \( B \) are *elementarily equivalent*, noted \( A \equiv B \) if they satisfy the same first-order \( \sigma \)-sentences, that is if \( A \equiv_n B \) for every \( n \).
Remark

Two \textbf{finite} elementary equivalent structures are isomorphic, but it is not usually the case for infinite structures.
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\[
\begin{align*}
\cdots & \\
\equiv & \\
\cdots & \\
\end{align*}
\]

Exercise
Prove that “there exists a path linking \( x \) and \( y \)” is not expressible by a first-order formula.
Locality

Definition

A formula $\phi$ with free variables $x_1, \ldots, x_p$ is $r$-local if, for every $G$ and every $v_1, \ldots, v_p \in G$ we have

$$G \models \phi(v_1, \ldots, v_p) \iff G\left[\bigcup_{i=1}^{p} N^r_G[v_i]\right] \models \phi(v_1, \ldots, v_p).$$

Example

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Gaifman Locality

Theorem (Gaifman 1982)

Every first-order formula $\psi(x_1, \ldots, x_n)$ is equivalent to a Boolean combination of $t$-local formulas $\chi(x_{i_1}, \ldots, x_{i_s})$ and basic local sentences of the form

$$\exists y_1 \ldots y_m \left( \bigwedge_{i=1}^{m} \phi(y_i) \land \bigwedge_{1 \leq i < j \leq m} \text{dist}(y_i, y_j) > 2r \right)$$

where $\phi$ is $r$-local. Furthermore $r \leq 7^{\text{qrank}(\psi)-1}$, $t \leq 7^{\text{qrank}(\psi)-1}/2$, $m \leq n + \text{qrank}(\psi)$, and, if $\psi$ is a sentence, only basic local sentences occur in the Boolean combination.
Exercises

Definition

A graph $G$ has the \textit{$n$-extension property} if for every disjoint subsets $A_0$ and $A_1$ of $G$ with $|A_0| + |A_1| \leq n$ we have:

$$\exists v \notin A_0 \cup A_1 \ (\forall u \in A_0 \ E(u, v) = 0) \land (\forall u \in A_1 \ E(u, v) = 1).$$

Exercise

If $G_1$ and $G_2$ have the \textit{$n$-extension property} then $G_1 \equiv_n G_2$.

Exercise (Quantifier elimination)

For every $\phi$ with $t$ free variables and $\text{qrank}(\phi) = r$ there exists quantifier free $q$ s.t. $\forall G$ with $(t + r - 1)$-extension property we have $G \models \phi \iff q$. 